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Geometrically invariant interactions in capillary wave surface models†

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Abstract. We examine the role of geometrically invariant interactions, such as curvature, in modulating the long-distance behaviour of surface tension models in the context of a perturbative ε-expansion approach first discussed by Wallace and Zia. A suggestive semiclassical calculation shows how some of these interactions arise. The renormalisation and, hence, scaling properties of the simplest of these interactions, the contraction of the second fundamental form with the metric tensor $(g \cdot b)$, are considered.

1. Introduction

Fluctuating surfaces and their implications for the critical properties of systems have been studied (Wallace and Zia 1979, Lowe and Wallace 1980, Lowe 1982). The simplest type of fluctuating surface (i.e. one which is $(d-1)$-dimensional in a d-bulk dimensional system) arises in Ising-like systems where it represents the interface between adjacent regions of 'up' and 'down' spins. In a field theoretic model, such an interface arises as a solution of the Euler–Lagrange equations which interpolates between field values at the bottom of two potential wells of equal depth.

Other authors have considered the interface profile (Jasnow and Rudnick 1978a,b, Ohta and Kawasaki 1977, Wallace and Zia 1979). Other generalisations have been studied (Lawrie and Lowe 1981, Lowe 1982) where an interface with an O(2) field defined on it has been considered.

The leading long-distance behaviour of the simple interface is governed by a Hamiltonian which is proportional to the area of the surface as its fluctuates into the remaining dth dimension. This result can be derived from a semiclassical calculation (Diehl *et al* 1980), which begins by considering the fluctuations about the interface-type solution of the Euler–Lagrange equations. A gapless (Goldstone) field, $f(y)$, where y is defined on a $(d-1)$-dimensional flat reference plane, arises because the solution breaks the Euclidean invariance of the system. This Goldstone mode can represent the position of the surface as it fluctuates with respect to the flat reference plane. One then proceeds, in principle, to integrate out all but the gapless mode in the partition function. This will leave us with an effective Hamiltonian, the most relevant (in the long-distance limit) interactions of which will be proportional to the surface area, involving only first derivatives of the field f. There will also be less relevant interactions involving higher derivatives of f. As the surface can be described by a field f, the derivatives of f can also be understood in geometric language. The interface

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in this case is a $(d-1)$-dimensional submanifold of a Euclidean d-dimensional manifold. The geometry of this $(d-1)$-dimensional submanifold is determined by g_{ij} and b_{ij}: they are coefficients of first and second fundamental forms, respectively. For the submanifold with the parametrisation

$$z + f(y^i) = 0 \qquad \text{where } i, j = 1, 2, \ldots, d-1, \tag{1.1}$$

g_{ij} and b_{ij} are given by

$$g_{ij} = \delta_{ij} + (\partial f/\partial y^i)\,\partial f/\partial y^j \tag{1.2}$$

$$b_{ij} = [1 + (\nabla f)^2]^{-1/2}\,\partial^2 f/\partial y^i\,\partial y^j. \tag{1.3}$$

The first fundamental form (metric tensor) can be illustrated by considering an infinitesimal unit of distance ($\mathrm{d}s$) on the curved submanifold which can be straightforwardly shown to be given by $(\mathrm{d}s)^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$. The second fundamental form governs the deviation of the surface from a tangent plane—i.e. the point on the surface given by the ordinates $y_i + dy_i$ from the tangent plane at y_i can be seen to be $\delta = \frac{1}{2}b_{ij}\,dy_i\,\mathrm{d}y_j$.

The higher-derivative interactions will take the form of geometric invariants such as $g \cdot b$ (the contraction of the second fundamental form with the metric tensor g_{ij}), curvature, $(g \cdot b)^2$, etc (Eisenhart 1926, Sternberg 1964).

In practice, it is highly difficult to perform the integration over the massive fields to all but the leading order, i.e. to obtain the surface area term. However, we present a highly suggestive semiclassical calculation which displays the origin of some of the higher-derivative interactions. The role of the $E(d)$ symmetry in determining the form of these interactions is considered (see Wallace 1981).

We then proceed to perform a perturbative analysis of the resulting effective Hamiltonian in $d = 1 + \varepsilon$ bulk dimensions (Wallace and Zia 1979) in a way which is similar to the analysis of the nonlinear sigma model in $(2 + \varepsilon)$ dimensions (Brézin and Zinn-Justin 1976). The higher-derivative interactions are treated as small perturbations to the surface tension term. An effective potential technique first developed by Forster and Gabriunas (1981) is extended and used to study the renormalisation properties of these interactions. We then proceed to demonstrate explicitly the renormalisability of the simplest of these interactions (i.e. $g \cdot b$) to one loop and hence derive its renormalisation group scaling properties.

2. Semiclassical calculation

The starting point for this calculation is a field theoretic model characterised by the Hamiltonian ($H(\phi)$) with a potential which possesses two minima of equal depth in d spatial dimensions. This model will possess a solution of the Euler–Lagrange equations which interpolates between the two minimum field values

$$\phi(x) = \phi_c(z) \tag{2.1}$$

where the solution depends on only one ordinate z.

As discussed previously (Lowe 1982), the technique for deriving the effective Hamiltonian for the Goldstone modes consists of finding a modification of the configuration (2.1) which allows for the Goldstone modes but is a solution of the Euler–

Lagrange equations to within acceptable errors, i.e. those which involve higher derivatives of the f field. In the partition function

$$Z = \int D[\phi] \exp(-H(\phi)), \tag{2.2}$$

we make the change of variables

$$\phi(x) = \phi_c(u) + \eta((z + f(y))/[1 + (\nabla f)^2]^{1/2}, y) \tag{2.3}$$

where

$$u = \frac{z + f(y)}{[1 + (\nabla f(y))^2]^{1/2}} + \frac{1}{2} \frac{(z + f(y))^2}{[1 + (\nabla f(y))^2]^{5/2}} \nabla_i f \nabla_j f \nabla_{ij} f \tag{2.4}$$

and y^i are the $d-1$ ordinates perpendicular to z. We must also impose

$$\int \frac{\mathrm{d}\phi_c(z)}{\mathrm{d}z} \eta(z, y) \, \mathrm{d}z = 0 \tag{2.5}$$

in order to avoid double counting of the Goldstone degrees of freedom.

This ensures that the change of variables (2.3) is unambiguous, and that the η field will contain no massless modes which would give rise to infrared divergences when loops in η propagators are considered.

On substitution of (2.3) into the functional integral (2.2), we obtain

$$Z = \int D[f(y)] D[\eta(z, y)]$$

$$\times \exp\left(\int \mathrm{d}^{d-1} y [1 + (\nabla f)^2]^{1/2} [1/T + cg \cdot b + dR] + \mathrm{O}(\eta^2) + \mathrm{O}(\eta) \right) \tag{2.6}$$

where

$$g \cdot b = \frac{\partial f}{\partial y^i} \frac{\partial^2 f}{\partial y^i \partial y^j} \frac{\partial f}{\partial y^j} [1 + (\nabla f)^2]^{-3/2} \tag{2.7}$$

$$R = \left[\frac{\partial^2 f}{\partial y^i \partial y^j} \frac{\partial^2 f}{\partial y^i \partial y^j} - \left(\frac{\partial^2 f}{\partial y^i \partial y^i} \right)^2 \right] [1 + (\nabla f)^2]^{-1} \tag{2.8}$$

$$1/T = \int \mathrm{d}z [\mathrm{d}\phi_c(z)/\mathrm{d}z]^2 \tag{2.9}$$

$$c = \int \mathrm{d}z \, z [\mathrm{d}\phi_c(z)/\mathrm{d}z]^2 \tag{2.10}$$

$$d = \int \mathrm{d}z \, z^2 [\mathrm{d}\phi_c(z)/\mathrm{d}z]^2 \tag{2.11}$$

and R is the curvature scalar. These expressions can be shown to be consistent with the formalism given by (1.1)–(1.3) up to total derivatives.

There will of course be a Jacobian factor which is in fact negligible from our point of view. Apart from any other consideration, the Jacobian will involve a factor of a one-dimensional spatial delta function which can be ignored in a dimensional regularised formalism (see Lawrie and Lowe 1981).

The key point is that the quadratic terms in η ensure that the η field is massive, while u has been carefully selected to ensure that the term linear in η has coefficients which involve third derivatives of f or the equivalent. The integration of the massive field can now be performed trivially to zero-loop (tree-diagram) level, since the extra f interactions so generated will be less relevant than the interactions explicitly in (2.6).

This must be treated with some caution because we must also evaluate the one- and two-loop terms in η, in order to obtain the full long-distance behaviour of the system to this order. However, the emergence of the curvature scalar is amusing.

In fact, there is a very limited number of interaction terms which possess the full $E(d)$ symmetry of the original system. Particularly restrictive are the symmetry transformations which rotate one of the y axes into the z axis. These are nonlinearly realised on the f field. In fact, only the geometric invariants can occur explicitly to the order to which we are working and they include $(g \cdot b)^2$ as well as the objects displayed in (2.6).

The absence of $(g \cdot b)^2$ in (2.6) is probably not significant since, in any case, $(g \cdot b)^2$ and R will presumably mix under renormalisation and hence under their renormalisation group flows. Unfortunately, the technical complexity of the calculations has prevented us from confirming this explicitly.

3. Renormalisation of $(g \cdot b)$

In this section we will consider the Hamiltonian

$$H_{\text{eff}}(f) = \int \mathrm{d}^{d-1}y\{[1+(\nabla f)^2]^{1/2}[1/T + cg \cdot b] + \tfrac{1}{2}m^2 f^2\}. \tag{3.1}$$

The mass term is introduced to control infrared problems. It is worth observing that the coefficient c will be zero in those systems in which the 'interface' solution is antisymmetric under reversal of the z axis, which in turn is a consequence of a '$\phi \to -\phi$' symmetry of the original Hamiltonian.

As has been demonstrated by a number of authors (Wallace and Zia 1979, Forster and Gabriunas 1981), the surface tension part of the Hamiltonian is perturbatively renormalisable in an ε-expansion context for $\varepsilon = d - 1$ with a single coupling constant (T) renormalisation. A renormalised dimensionless coupling constant t is introduced,

$$T = Z\kappa^{-\varepsilon}t, \tag{3.2}$$

and κ is a renormalisation mass scale.

A beta function can be derived from this coupling constant renormalisation. This has a zero for $t = \mathrm{O}(\varepsilon)$ of the kind which indicates a second-order phase transition. A renormalisation group invariant length scale can be introduced which diverges as this fixed point value of t is approached from below.

Let us consider $cg \cdot b$ to be a small perturbation. The point at issue here is to demonstrate that, provided we restrict ourselves to diagrams with only one vertex coming from the expansion of $g \cdot b[1+(\nabla f)^2]^{1/2}$ in powers of f (i.e. those terms which are linear in C), $g \cdot b[1+(\nabla f)^2]^{1/2}$ will be multiplicatively renormalisable. In practice, if only because of the infinite set of diagrams one would have to consider, it is best to use the free energy functional (effective potential) method of Forster and Gabriunas (1981).

This involves calculating the free energy for the system (3.1) for certain configurations. In more conventional theories (such as ϕ^4 in four dimensions), where the zero-momentum parts of vertices are of interest because these contain all the divergences, we evaluate the free energy functional for spatially independent field configurations. By contrast, the divergent parts of an n-point diagram involving one insertion from $g \cdot b[1+(\nabla f)^2]^{1/2}$ and any number of insertions from the $[1+(\nabla f)^2]^{1/2}$ (surface tension) part of the Hamiltonian is proportional to k^{n+1} where k is a typical external momentum scale. This is of course due to the presence of one second derivative of the field f and any number of first derivatives of f. It can be shown that we can study this part of the theory by evaluating the free energy functional for a configuration of f involving terms linear and quadratic in y (i.e. with first and second derivatives only),

$$f(y) = N_i y_i + \tfrac{1}{2} M_{ij} y_i y_j, \tag{3.3}$$

and working to first order in M and any power of N (see Forster and Gabriunas 1981).

The calculation presents some interesting technical aspects. In particular, a number of the vertices involved in the calculation depend linearly on y_i. The calculation is performed in momentum space and y_i is interpreted as a Fourier transformation of a derivative of a momentum conservation delta function

$$y_i = \mathrm{i} \int \frac{\mathrm{d}^\epsilon p}{(2\pi)^\epsilon} \frac{\partial}{\partial p_i} \delta(p)\, \mathrm{e}^{\mathrm{i}p \cdot y}. \tag{3.4}$$

After a long and tedious calculation, the counterterm necessary to render the system finite transpires to be proportional to $g \cdot b[1+(\nabla f)^2]^{1/2}$. A simple multiplicative renormalisation of the coefficient c thus suffices to make the system finite,

$$c^{\mathrm{R}} = z_c^{-1} c^b \qquad z_c = (1 + t/\epsilon). \tag{3.5}$$

We are now in a position to calculate the renormalisation group flow of the renormalised c^{R}. For temperatures close to the critical value,

$$c^{\mathrm{R}} \sim \kappa^{1-\epsilon+\mathrm{O}(\epsilon^2)}. \tag{3.6}$$

The picture of a long-distance behaviour of the Hamiltonian (3.1) governed by the fixed point of the surface tension Hamiltonian modified by corrections due to higher-derivative geometric invariants is thus confirmed to be stable under renormalisation group flows.

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